

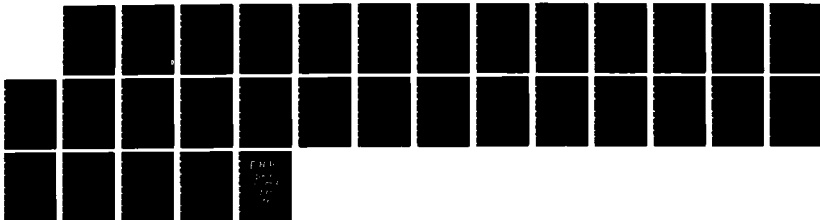
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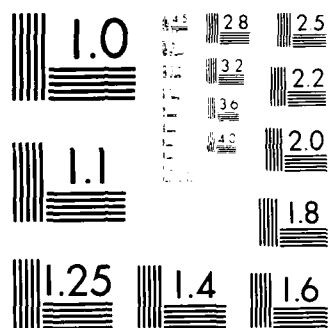
WEAK CONVERGENCE OF SUMS OF MOVING AVERAGES IN THE  
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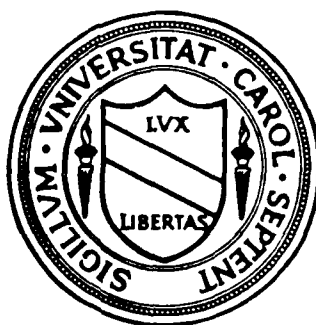
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WEAK CONVERGENCE OF SUMS OF MOVING  
AVERAGES IN THE  $\alpha$ -STABLE DOMAIN OF ATTRACTION

by

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Technical Report No. 191

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Weak Convergence of Sums of Moving  
Averages in the  $\alpha$ -Stable Domain of Attraction

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Abstract

Skorohod has shown that the convergence of sums of i.i.d. random variables to an  $\alpha$ -stable Levy process, with  $0 < \alpha < 2$ , holds in the weak  $J_1$  sense. We show that for sums of moving averages with at least 2 non-zero coefficients, weak  $J_1$  convergence cannot hold, however, if the moving average coefficients are positive, weak  $M_1$  convergence usually does hold.

Key words and phrases:  $\alpha$ -stable, weak Skorohod convergence, moving averages, bisection method.

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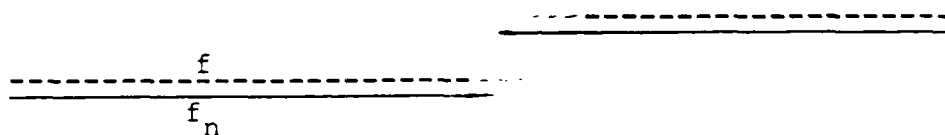
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## Introduction

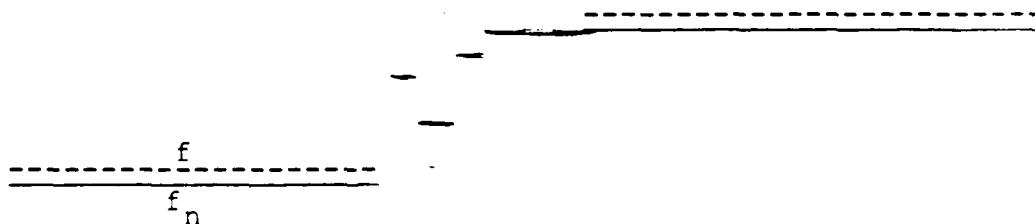
The investigation of functional limit theorems for processes with paths in  $D[0,1)$  (functions continuous to the right and with left limits to the left) was started by Skorohod (1956). In that paper, Skorohod introduced four topologies in  $D[0,1)$ , called  $J_1, J_2, M_1, M_2$ .

The four topologies differ in the way converging sequences of functions  $f_n$  (deterministic functions) are required to approach their limit  $f$  in the neighborhood of a jump of  $f$ . We indicate now roughly what these ways of approaching a jump are.

In the case of the  $J_1$  topology,  $f_n$  have to have one jump only (which will approximate in location and height the jump of  $f$ ).



In the case of the  $M_2$  topology,  $f_n$  are allowed to jump several times through intermediary values falling roughly in between the left and right limits of  $f$ .



In the case of the  $M_1$  topology, several jumps are allowed, but they have to go roughly in the same direction, like stairs (which get "compressed" into a single jump of  $f$ ).

$$\frac{f_n}{f}$$

$J_1$  convergence is thus appropriate only when one jump of the limit comes out of a single jump in  $f_n$ . This turned out to be the case for the normalized sums of i.i.d. random variables in the  $D(\alpha)$ -domain of attraction, with  $0 < \alpha < 2$ ; in this case, as shown by Skorohod (1957), weak  $J_1$  convergence holds.

We will show, however, that in the case of normalized sums of moving averages of i.i.d.'s in  $D(\alpha)$ , with at least two non zero coefficients, weak  $J_1$  convergence does not hold (with only one non zero coefficient it does hold, by Skorohod's result). The reason is that in this case, one jump of the limit comes out of "stairs" with at least two steps. If we assume also  $c_1 \geq 0$ , then we can prove, however, that weak  $M_1$  convergence holds (since the steps will go then in the same direction).

# 1. Statement of Results:

We now introduce some notation and state the main results:

Let  $X_i$  be an i.i.d. sequence belonging to  $D(\alpha)$ ,  $0 < \alpha < 2$ . We assume also  $EX_i = 0$  when  $1 < \alpha < 2$ .

Let  $c_i, i \in \mathbb{Z}$ , be a sequence such that there exists  $\nu, \nu < \alpha$ , so that

$$(1.1) \quad \sum_i |c_i|^\nu < \infty.$$

(1.1) ensures that the moving averages:

$$(1.2) \quad Y_m = \sum_{i=-\infty}^{\infty} c_{m-i} X_i$$

are well defined in  $L^\nu$  sense (and in fact also in a.s. sense, cf. Kawata (1972), Theorems 12.11.2, 12.10.4).

Let  $a_n$  be constants such that

$$(1.3) \quad \sum_{i=1}^{[nt]} X_i / a_n \xrightarrow{\text{f.d.d.}} X_\alpha(t),$$

where  $X_\alpha(t)$  is a Levy  $\alpha$ -stable process, and  $\xrightarrow{\text{f.d.d.}}$  denotes convergence of the finite-dimensional distributions.

In the sequel, we assume also that  $\nu$  in 1.1 satisfies  $\nu \leq 1$ . In this case, Davis and Resnick (1984), and Astrauskas (1983) have shown that the normalized sums of  $Y_i$  are attracted to a Levy  $\alpha$ -stable process as well, namely:

Theorem 1 (Davis and Resnick (1984), Theorem 4.1, Astrauskas (1983), Theorem 1i: When  $\sum |c_i| < \infty$ ,

$$(1.4) \quad \sum_{i=1}^{[nt]} Y_i / a_n \xrightarrow{\text{f.d.d.}} \left( \sum_i c_i \right) X_\alpha(t)$$

holds, where  $a_n$  and  $X_\alpha(t)$  are the same as in the case of independent summands (1.3).

Can the f.d.d. convergence in (1.4) be replaced by weak convergence in  $D[0,1)$  with respect to one of the Skorohod topologies? We show that weak  $J_1$  convergence can not hold, if the moving coverage has at least 2 non zero coefficients.

Theorem 2: Suppose that  $c_0 \neq 0$  and  $c_{i_0} \neq 0$  for some  $i_0 \neq 0$ . Suppose also  $c_i = 0$  for  $i < 0$  and  $i > K$ , for some finite  $K$ . Then, convergence in (1.4) does not hold in  $w(J_1)$  sense.

Remark: 1) The assumption that only finitely many  $c_i$ 's are different from 0 could probably be removed.

2) When only one coefficient  $c_i$  is non zero (when the summands are independent),  $w(J_1)$  convergence does hold, by Skorohod (1957).

However, under some extra assumptions, weak  $M_1$  convergence holds.

Theorem 3: Suppose that  $c_i \geq 0$ , and that either

a)  $\alpha \leq 1$

or

b)  $\alpha > 1$ , and (T.C.) holds

then, weak  $M_1$  convergence holds in (1.4).

Note: (T.C.) is a weak technical condition on the sequence  $c_i$ , which could probably be removed, and will be specified below.

As far as  $M_2$  convergence, we make a

Conjecture: If  $c_i = 0$  for  $i \leq 0$ , and for every  $K$ ,

$$0 \leq \sum_{i=1}^K c_i / \sum_{i=1}^{\infty} c_i \leq 1,$$

then weak  $M_2$  convergence in (1.4) holds.

We will give now a heuristic explanation of our results.  
First, let us assume  $Y_i$  are finite moving averages:

$$Y_i = \sum_{j=0}^K c_j X_{i-j}$$

Heuristically, most of the sequence  $X_{i,n} = X_i/a_n \approx 0$  (is negligible), except for a sequence of "big values",  $X_{i_0,n}, X_{i_1,n}, \dots, X_{i_k,n}$ , which are spread apart:

$$i_0 \ll i_1 \ll \dots \ll i_k \ll \dots$$

It follows that most of  $Y_{i,n} = Y_i/a_n$  are also negligible; however, a big value  $X_{i_0,n}$  produces  $K+1$  big values in the sequence  $Y_{i,n}$ :

$$Y_{i_0,n} \approx c_0 X_{i_0,n}$$

$$Y_{i_0+1,n} \approx c_1 X_{i_0,n}$$

$$Y_{i_0+k,n} \approx c_k X_{i_0,n}$$

Thus, the  $Y_{i,n}$  sequence increases by "stairs", covering  $\frac{K}{n} \rightarrow 0$  on the  $x$  axis, and thus converging to a single jump in the limit. From the heuristics given in the Introduction, we see that:

If the stairs have at least two steps, we can't have  $J_1$  convergence (Theorem 2).

If the stairs go the same direction, we have  $M_1$  convergence (Theorem 3).

If the stairs fall in between the lowest level and the highest one, we might have  $M_2$  convergence (Conjecture).

Additional Remarks:

1. Some conditions on the  $c_i$  are necessary, if we are to get any weak Skorohod convergence at all. Indeed, consider the example:

$$c_0 = 1, c_1 = -1, c_k = 0 \text{ for } k \neq 0, 1.$$

Here,

$$\frac{1}{a_n} \sum_{i=1}^{[nt]} Y_i = \frac{X_{[nt]} - X_0}{a_n} \text{ f.d.d. } > 0$$

but f.d.d. convergence cannot be replaced by weak convergence in any of the four topologies, because, as is widely known,

$\sup_{0 \leq t \leq 1} X_{[nt]}/a_n$  converges in distribution to a non-zero limit, and  $\sup_{0 \leq t \leq 1}$  is a continuous functional in all the four Skorohod topologies.

2. On the other hand, if we make the strong assumptions  $c_i \geq 0, X_i \geq 0$ , (these assumptions are compatible with  $\alpha < 1$ ), then since  $\sum_{i=1}^{[nt]} Y_i/a_n$  has monotone path, weak  $M_1$  convergence holds automatically. Thus, in this case, with no work, we get weak convergence in  $M_1$  sense.

We describe now the condition (T.C.) of Theorem 3 (which is necessary only for  $\alpha > 1$ ). For  $\alpha' > 1 \geq \nu$ , let:

$$(1.5) \quad s(\alpha', c.) = \left( \sum_i |c_i|^\nu \right) \left( \sum_i |c_i| \right)^{\alpha' - \nu}$$

Note that if  $1 = \nu$ , then  $s(\alpha', c.) = \left( \sum_i |c_i| \right)^{\alpha'}$ .

Let

$$(1.6a) \quad c_i^{>m} = \begin{cases} c_i & \text{for } |i| > m \\ 0 & \text{otherwise} \end{cases}$$

$$(1.6b) \quad c_i^{\leq m} = c_i - c_i^{> m}$$

The condition on  $c_i$  is:

$$(T.C.) \quad \lim_{n \rightarrow \infty} s(\alpha - \eta, c_i^{> n}) (\ln n)^{1+\alpha+n}$$

for some  $\eta > 0$  small enough.

(T.C.) is satisfied in lots of cases of interest, like for example when  $s(\alpha - \eta, c_i^{> n})$  is dominated by a regularly varying sequence with strictly negative exponent. We will show in fact:

Lemma 1: a) If there exists  $\nu < 1$  as required (i.e., satisfying  $\sum_i |c_i|^\nu < \infty$ ,  $\nu < \alpha$ ), and  $c_i$  is a monotone sequence, then (T.C.) holds.

b) If  $\sum_i |c_i| < \infty$ , but  $\sum_i |c_i|^{\nu'} = \infty$ ,  $\forall \nu' < 1$ , then (T.C.) might not hold, even if  $c_i$  is a monotone sequence.

The paper is organized as follows: in Section 2 we state the main steps leading to the proof of Theorem 3, and prove Theorem 3. All the other proofs are contained in Section 3.

## 2. Proof of Theorem 3: Let:

$$(2.1) \quad J(x_1, x_2, x_3) = \min \{ |x_2 - x_1|, |x_3 - x_2| \}$$

$$(2.2) \quad M(x_1, x_2, x_3) = \text{the distance from } x_2 \text{ to } [x_1, x_3] \\ = \begin{cases} 0 & \text{if } x_2 \in [x_1, x_3] \\ J(x_1, x_2, x_3) & \text{otherwise} \end{cases}$$

Let now  $H$  stand for either  $J$  or  $M$ , and introduce

the oscillation of a function  $Z(t)$  by:

$$(2.3) \quad w_{\delta}^H(Z) = \sup_{\substack{0 \leq t-a \leq \delta/2 \\ 0 \leq b-t \leq \delta/2}} H(Z(a), Z(t), Z(b))$$

For a definition of the Skorohod topologies, and their analysis, see Skorohod (1956).

Here we will need only a corollary of his Theorem 3.2.1:

Proposition 1: (Skorohod (1956)): Let  $Z_n(t)$  be processes in  $D[0,1]$  whose finite dimensional distributions converge to those of a process  $Z(t)$ . Let  $H$  stand for either  $J$  or  $M$ . Then, weak  $H_1$  convergence holds iff for every  $\varepsilon > 0$ .

$$(2.4) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\{w_{\delta}^H(Z_n) > \varepsilon\} = 0$$

Theorem 3 will be established by showing that in the case  $H = M$ , (2.4) holds. This is accomplished by considering first the case of finite moving averages, and, in the case of infinite moving averages, by truncation.

Let:

$$(2.5) \quad Z_n(t) = \sum_{i=1}^{[nt]} Y_i / a_n$$

By using the technique of Billingsley applied to the case of the  $M$ -topology (see Avram and Taqqu (1986)), it is enough to establish estimates uniform in  $n$  for:

$$(2.6) \quad M_n(a, t, b) := M(Z_n(a), Z_n(t), Z_n(b))$$

Proposition 2: If  $c_i \geq 0$ , and  $c_i = 0$  when  $|i| > K$ , for some finite  $K$ , then, for  $\eta > 0$  and small enough, and for  $n$  satisfying:

$$(2.7) \quad n^{\left(\frac{1}{2}-\eta\right)/(1+\alpha-\eta)} > K,$$

there exist a constant  $L$  independent of  $n$ , so that:

a)

$$(2.8) \quad P\{M_n(a,t,b) > \varepsilon\} \leq L\varepsilon^{-2(\alpha+\eta)}(b-a)^{1+2\eta}$$

b) Furthermore, there exists a constant  $k$  independent of  $n$  so that

$$(2.9) \quad P\{w_{\delta}^M(Z_n) > \varepsilon\} \leq Lk\varepsilon^{-2(\alpha+\eta)}\delta^{2\eta}$$

Convention: Here and from now on, a "constant" will mean a quantity which might depend on the distribution of the sequence  $Z_n$ , and on  $\alpha$  and  $\eta$ , but not on  $\delta, \varepsilon, a, b$  or  $n$ .

Note: 1) Part b) is an immediate corollary of Part a) and Theorem 1 of Avram and Taqqu (1986).

2) As a corollary of Skorohod's proposition, Theorem 1 and Proposition 2, it follows that Theorem 3 holds when  $Y_i$  are finite moving averages.

In the general case, we will decompose  $Y_i$ . We use the following notation:

Let  $K_n$  be a sequence converging to  $\infty$ ,

$$(2.10a) \quad c_i^{\leq K_n} = \begin{cases} c_i & \text{if } |i| \leq K_n \\ 0 & \text{otherwise} \end{cases}$$

and

$$(2.10b) \quad c_i^{>K_n} = c_i - c_i^{\leq K_n} = \begin{cases} 0 & \text{if } |i| \leq K_n \\ c_i & \text{if } |i| > K_n. \end{cases}$$

We let  $Y_{i,n}^{\leq K_n}, Y_{i,n}^{>K_n}$  be the moving averages with

coefficients  $c_i^{\leq K_n}, c_i^{> K_n}$  respectively, and their sums up

to  $[nt]$  will be  $z_n^{\leq K_n}(t), z_n^{> K_n}(t)$ .

$z_n^{\leq K_n}$  are sums of finite moving averages, to which Proposition 2 applies, while  $z_n^{> K_n}$  are sums of moving averages with "small" coefficients. They will be handled by the use of the following:

Proposition 3: Let  $z_n$  be defined as in 2.5, with  $y_i$  defined as in (1.2), but with sequences of coefficients  $c_i^{(n)}$  replacing the fixed sequence  $c_i$ . For  $\eta > 0$  small enough, there exist then constants  $L', k'$ , independent of  $n$ , such that:

a)

$$(2.11) \quad P(|z_n(t_2) - z_n(t_1)| > \varepsilon) \leq L' \varepsilon^{-(\alpha+\eta)} (t_2 - t_1) s(\alpha-\eta, c^{(n)}),$$

where

$$(2.12) \quad s(\alpha'; c) = \begin{cases} \sum_i |c_i|^{\alpha'} & \text{if } \alpha' \leq 1 \\ (\sum_i |c_i|^v) (\sum_i |c_i|)^{\alpha'-v} & \text{if } \alpha' > 1 \geq v \end{cases}$$

b)

$$P \left\{ \sup_{0 \leq t \leq 1} |z_n(t)| > \varepsilon \right\} \leq \begin{cases} L' k' \varepsilon^{-(\alpha+\eta)} s(\alpha-\eta, c^{(n)}) & \text{if } \alpha \leq 1 \\ L' k' \varepsilon^{-(\alpha+\eta)} (\ln n)^{1+\alpha+\eta} s(\alpha-\eta, c^{(n)}) & \text{if } \alpha > 1. \end{cases}$$

Proof of Theorem 3: We look for a sequence  $K_n$  small enough [satisfying (2.7)] so that Proposition 2 can be applied to the process  $z_n^{\leq K_n}$ , but big enough so that the estimate of

$P\left\{\sup_{0 \leq t \leq 1} |Z_n|^{>K_n}(t) > \varepsilon\right\}$  given in Proposition 3b), namely

$$(2.13) \quad e_n = \begin{cases} s(\alpha-\eta, c. >K_n) & \text{if } \alpha \leq 1 \\ (\ln_2 n)^{1+\alpha+\eta} s(\alpha-\eta, c. >K_n), & \text{if } \alpha > 1 \end{cases}$$

tends to 0, as  $n \rightarrow \infty$ . An adequate sequence is  $K_n = n^{1/6}$ . This  $K_n$  satisfies (2.7) since for  $\eta$  small enough  $n^{1/6} < n^{(1/2-\eta)/(1+\alpha-\eta)}$ . When  $\alpha \leq 1$   $e_n$  tends to 0, since  $K_n \rightarrow \infty$ . On the other hand, if  $\alpha > 1$ , by assumption (T.C.) of Theorem 3,

$$(2.14) \quad \lim_{n \rightarrow \infty} s(\alpha-\eta, c. \geq K_n) (\ln K_n)^{1+\alpha+\eta} = 0$$

$$= \frac{1}{6} \lim_{n \rightarrow \infty} s(\alpha-\eta, c. \geq K_n) (\ln n)^{1+\alpha+\eta}$$

and thus the estimate  $e_n$  tends to 0. Now it remains only to note that if  $w_{\delta}^M(Z_n^{\leq K_n}) \leq \frac{\varepsilon}{2}$ , and  $\sup_{0 < t \leq 1} |Z_n^{>K_n}(t)| \leq \frac{\varepsilon}{2}$ , then  $w_{\delta}^M(Z_n) \leq \varepsilon$ .

Thus

$$P\left\{w_{\delta}^M(Z_n) > \varepsilon\right\} \leq P\left\{w_{\delta}^M(Z_n^{\leq K_n}) > \frac{\varepsilon}{2}\right\} + P\left\{\sup_{0 \leq t \leq 1} |Z_n^{>K_n}(t)| > \frac{\varepsilon}{2}\right\}$$

$$\leq Lk\left(\frac{\varepsilon}{2}\right)^{-2(\alpha+\eta)} \delta^{2\eta} + L'k'\left(\frac{\varepsilon}{2}\right)^{-(\alpha+\eta)} e_n$$

(by Propositions 2,3).

Hence,

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\left\{w_{\delta}^M(Z_n) > \varepsilon\right\} \leq \lim_{\delta \rightarrow 0} Lk\left(\frac{\varepsilon}{2}\right)^{-2(\alpha+\eta)} \delta^{2\eta} = 0$$

Hence, by Theorem 1 and the Proposition of Skorohod (1956), Theorem 3 holds.

3. AUXILIARY RESULTS AND PROOFS

Proof of Theorem 2. Assume for convenience that

$c_0 > 0$  and  $|c_1| > c_0$ . Let

$$A_{n,\epsilon} = \left\{ \exists k, k \in \{1-K, \dots, n+K\} \text{ such that } X_{k,n} > \frac{\epsilon}{c_0} \right\}$$

$$A_{n,\epsilon}^{(1)} = \left\{ \exists k_1, k_2 \in \{1-K, \dots, n+K\}, k_1 \neq k_2 \text{ such that:} \right.$$

$$\left. X_{k_1,n} > \frac{\epsilon}{c_0}, |X_{k_2,n}| > \frac{\epsilon}{2 \sum_{i=0}^K |c_i|} \text{ and } |k_1 - k_2| \leq K \right\}.$$

We have  $A_{n,\epsilon}^{(1)} \subset A_{n,\epsilon}$ . On one hand,  $\forall \epsilon, \lim_{n \rightarrow \infty} P(A_{n,\epsilon}) \neq 0$ .

(It is well known that  $w - \lim_{n \rightarrow \infty} \max_{1 \leq k \leq n} X_{k,n} \neq 0$ , in fact

the limiting distribution of  $X_{k,n}$  is known.) On the other hand, since  $k_1$  can take at most  $n + 2K$  values,  $k_2$  can take at most  $K$  values (or conversely), we have

$$P(A_{n,\epsilon}^{(1)}) \leq 2(n+2K)K P\left\{|X_{k,n}| > \frac{\epsilon}{2}\right\} \cdot P\left\{|X_{k,n}| > \frac{\epsilon}{c_0}\right\}.$$

This tends to 0 as  $n \rightarrow \infty$  because  $X_{k,n} = X_{k/a_n}$  and from the definition of  $a_n$ , one has

$$P\left\{|X_{k,n}| > a\right\} = O\left(\frac{1}{n}\right), \forall a.$$

Therefore

$$(3.1) \quad \lim_{n \rightarrow \infty} P(A_{n,\epsilon} - A_{n,\epsilon}^{(1)}) > 0.$$

Note, now, that

$$(A_{n,\epsilon} - A_{n,\epsilon}^{(1)}) \text{ implies that } \left\{w_{1/n,n}^J(Z_n) > \epsilon/2\right\}.$$

Indeed, let  $i_*$  be such that

$$X_{i_*} = \max_{1 \leq i \leq n} X_i.$$

On  $A_{n,\epsilon} - A_{n,\epsilon}^{(1)}$  we have

$$Y_{i_*,n} = c_0 X_{i_*,n} + \sum_{j=1}^K c_j X_{i_*-j,n} > \frac{\epsilon}{2},$$

since

$$c_0 X_{1*,n} > \epsilon,$$

and

$$\left| \sum_{j=1}^K c_j X_{1*-j,n} \right| \leq \frac{\epsilon}{2 \sum_{j=0}^K |c_j|} \left( \sum_{j=1}^K |c_j| \right) \leq \frac{\epsilon}{2}.$$

Similarly, on  $A_{n,\epsilon} - A_{n,\epsilon}^{(1)}$  we have:

$$\begin{aligned} \left| Y_{1*+1,n} \right| &= \left| c_1 X_{1*-j,n} + \sum_{\substack{j=0 \\ j \neq 1}}^K c_j X_{1*+1-j,n} \right| \\ &> \frac{|c_1|}{c_0} c_0 X_{1*,n} - \frac{\epsilon}{2} > \frac{\epsilon}{2} \end{aligned}$$

and thus:

$$A_{n,\epsilon} - A_{n,\epsilon}^{(1)} \text{ implies that } \left\{ |Y_{1*,n}| > \frac{\epsilon}{2}, |Y_{1*+1,n}| > \frac{\epsilon}{2} \right\}$$

which in turn implies  $\left\{ w_{1/n,n}^J(Z_n) > \frac{\epsilon}{2} \right\}$ .

It follows from (3.1), then, that

$$\lim_{n \rightarrow \infty} P \left\{ w_{1/n,n}^J(Z_n) > \frac{\epsilon}{2} \right\} > 0,$$

and thus we cannot have

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} P \left\{ w_{\delta,n}^J(Z_n) > \frac{\epsilon}{2} \right\} = 0,$$

which is necessary for  $J_1$  convergence.  $\square$

Remark: If in the preceding proof all the  $c_i$  are non-negative, then  $Y_{1*,n} > \frac{\epsilon}{2}$ ,  $Y_{1*+1,n} > \frac{\epsilon}{2}$ ; though the two consecutive changes of  $Z_n$  are big in absolute value, they "go" in the same direction and thus produce zero  $M_1$  oscillation.

From now on, we will assume w.l.o.g. that  $\sum_{i=1}^K c_i = 1$  (and thus  $\sum_{i=1}^K c_i < 1$ , and  $c_1 < 1$ ).

Proof of Lemma 1: Choose  $\epsilon > 0$  so that:  $\epsilon + \epsilon \leq 1 \leq \epsilon + \epsilon$ ; then,

$$\begin{aligned}
s(\alpha - \eta, c, \geq n) &= \left( \sum_{|i| \geq n} |c_i|^\nu \right) \left( \sum_{|i| \geq n} |c_i| \right)^{\alpha - \eta - \nu} \leq \\
&\leq \left[ |c_n|^\eta \sum_{|i| \geq n} |c_i|^{1-\eta} \right]^{\alpha - \eta - \nu} \leq \left[ |c_n|^\eta \sum_{|i| \geq n} |c_i|^\nu \right]^\eta \leq \\
&\leq (n|c_n|)^\eta n^{-\eta^2} = o(n^{-\eta^2}).
\end{aligned}$$

Thus,  $s(\alpha - \eta, c, \geq n)$  is bounded by a negative power of  $n$ , and (T.C.) is satisfied.

b) Consider the following counter example

$$c_i = \frac{1}{|i|(\ell n |i|)^{1+\beta}}, \quad 0 < \beta \leq 1 + \frac{1}{\alpha}.$$

Here,  $\sum_{|i| \geq n} c_i = O\left(\frac{1}{(\ell n - n)^\beta}\right).$

Thus,  $\sum_1 |c_i| < \infty$ , but  $\sum_1 |c_i|^{1-\eta} = \infty, \forall \eta > 0$ .

Then, if  $\alpha - \eta > 1$ , we have:

$$s(\alpha - \eta, c, \geq n) = \left( \sum_{|i| \geq n} c_i \right)^{\alpha - \eta} = O\left(\frac{1}{(\ell n - n)^{\beta(\alpha - \eta)}}\right).$$

To satisfy (T.C.), we have to find, then,  $\eta > 0$ , such that  $\beta(\alpha - \eta) > 1 + \alpha + \eta$ ; that is,  $\eta\beta + \eta < \beta\alpha - (1 + \alpha)$ .

This is possible iff  $\beta\alpha - (1 + \alpha) > 0$ , i.e., iff  $\beta > 1 + \frac{1}{\alpha}$ . Since  $\beta \leq 1 + \frac{1}{\alpha}$ , (T.C.) cannot be

satisfied. □

We turn now to some auxiliary results. The first is a classical result, often used when dealing with r.v.'s in  $D(\alpha)$ .

Lemma 2: Let  $X_{i,n} = X_i/a_n$ , where  $X_i$  is an i.i.d. sequence in  $D(\alpha)$ , where either:  $0 < \alpha \leq 1$ , or  $1 < \alpha < 2$  and  $E X_i = 0$ , and where  $a_n$  are the normalization constants in the C.L.T. Let  $|b_{i,n}| < 1, n, i = 1, 2, \dots$

and let  $\varepsilon > 0$ . Then, if  $n > 0$  is small enough, there exists a constant  $M$  depending on the distribution of  $X_1$ , but not on  $m$  or  $\varepsilon$ , such that

$$(3.2) \quad P\left\{\sup_{1 \leq k \leq m} \left| \sum_{i=1}^k b_{i,n} X_{i,n} \right| \geq \varepsilon\right\} \leq M \frac{\varepsilon^{-(\alpha+\eta)}}{n} \sum_{i=1}^m |b_{i,n}|^{\alpha+\eta},$$

for all  $m$ , even  $m = \infty$ .

Proof of Lemma 2. If  $\alpha < 1$ , we let

$$X_{i,n}^{\leq} = X_{i,n} \mathbf{1}(|X_{i,n}| \leq 1)$$

$$X_{i,n}^{>} = X_{i,n} \mathbf{1}(|X_{i,n}| > 1)$$

Let  $\eta > 0$  be such that  $\alpha - \eta > 0$ ,  $\alpha + \eta < 1$ , and consider

$$\begin{aligned} (3.3) \quad & P\left\{\sup_{1 \leq k \leq m} \left| \sum_{i=1}^k b_{i,n} X_{i,n}^{>} \right| \geq \frac{\varepsilon}{2}\right\} \leq \\ & P\left\{\sum_{i=1}^m |b_{i,n}| |X_{i,n}^{>}| \geq \frac{\varepsilon}{2}\right\} \leq \\ & \leq \left(\frac{\varepsilon}{2}\right)^{-(\alpha-\eta)} E\left(\sum_{i=1}^m |b_{i,n}| |X_{i,n}^{>}|^{\alpha-\eta}\right) \leq \\ & \leq \frac{\varepsilon^{-(\alpha-\eta)}}{2} \sum_{i=1}^m |b_{i,n}|^{\alpha-\eta} E|X_{i,n}^{>}|^{\alpha-\eta}. \end{aligned}$$

Similarly,

$$\begin{aligned} (3.4) \quad & P\left\{\sup_{1 \leq k \leq m} \left| \sum_{i=1}^k b_{i,n} X_{i,n}^{\leq} \right| \geq \frac{\varepsilon}{2}\right\} \leq \\ & \leq \left(\frac{\varepsilon}{2}\right)^{-(\alpha+\eta)} E|X_{1,n}^{\leq}|^{\alpha+\eta} \sum_{i=1}^m |b_{i,n}|^{\alpha+\eta}. \end{aligned}$$

Let

$$M' = \sup_n \left\{ n E|X_{1,n}^{>}|^{\alpha-\eta} \right\} \vee \sup_n \left\{ n E|X_{1,n}^{\leq}|^{\alpha+\eta} \right\}.$$

Since  $M' < \infty$  [see Astrauskas (1983), Lemma 1], we see that (3.3), (3.4) imply (3.2), with  $M = 2^{1+\alpha+\eta} M'$ , whether  $m$  is finite or not.

When  $\alpha > 1$ , let  $\eta$  be such that  $\alpha - \eta \geq 1$ , and let

$$(3.5) \quad \begin{aligned} \bar{X}_{1,n}^{\leq} &= X_{1,n}^{\leq} - E X_{1,n}^{\leq} \\ \bar{X}_{1,n}^{\geq} &= X_{1,n}^{\geq} + E X_{1,n}^{\leq}. \end{aligned}$$

Thus,  $E(\bar{X}_{1,n}^{\leq}) = 0$ , and since  $E X_{1,n} = 0$ , we have

$E(\bar{X}_{1,n}^{\geq}) = 0$  also.

We will show that (3.3), (3.4) hold again, with

$\bar{X}_{1,n}^{\geq}, \bar{X}_{1,n}^{\leq}$  replacing  $X_{1,n}^{\geq}, X_{1,n}^{\leq}$ . Note first that  $\sum_{i=1}^k b_{i,n} \bar{X}_{1,n}^{\leq}$  and  $\sum_{i=1}^k b_{i,n} \bar{X}_{1,n}^{\geq}$  are martingales (as  $k$  varies).

Using the maximal inequality:

$$P\left\{\sup_{1 \leq k \leq m} |S_n| \geq \lambda\right\} \leq \lambda^{-p} E|S_m|^p,$$

which holds for  $p \geq 1$ , and  $S_n$  a martingale [see Shiryaev (1979), page 464], and the Bahr-Essen inequality (Lemma 1 of the previous chapter), we have:

$$\begin{aligned} (3.3') \quad P\left\{\sup_{1 \leq k \leq m} \left| \sum_{i=1}^k b_{i,n} \bar{X}_{1,n}^{\geq} \right| \geq \frac{\epsilon}{2}\right\} \\ \leq \left(\frac{\epsilon}{2}\right)^{-(\alpha-\eta)} E \left| \sum_{i=1}^m b_{i,n} \bar{X}_{1,n}^{\geq} \right|^{\alpha-\eta} \\ \leq \left(\frac{\epsilon}{2}\right)^{-(\alpha-\eta)} \sum_{i=1}^m |b_{i,n}|^{\alpha-\eta} E \left| \bar{X}_{1,n}^{\geq} \right|^{\alpha-\eta}, \end{aligned}$$

and, similarly,

$$(3.4') \quad P\left\{ \sup_{1 \leq k \leq m} \left| \sum_{i=1}^k b_{i,n} \bar{X}_{i,n}^{\leq} \right| \geq \frac{\epsilon}{2} \right\} \\ \leq \frac{\epsilon^{-(\alpha+\eta)}}{2} \sum_{i=1}^m |b_{i,n}|^{\alpha+\eta} E \left| \bar{X}_{i,n}^{\leq} \right|^{\alpha+\eta}.$$

Since, by Jensen's inequality,

$$E |X_{i,n}^{\leq} - E X_{i,n}^{\leq}|^{\alpha+\eta} \leq 2^{\alpha+\eta-1} (E |X_{i,n}^{\leq}|^{\alpha+\eta} + |E X_{i,n}^{\leq}|^{\alpha+\eta}) \\ \leq 2^{\alpha+\eta} E |X_{i,n}^{\leq}|^{\alpha+\eta},$$

and similarly,

$$E |X_{i,n}^{>} - E X_{i,n}^{>}|^{\alpha-\eta} \leq 2^{\alpha-\eta} E |X_{i,n}^{>}|^{\alpha-\eta},$$

we see that:

$$M'' = \sup_n (n E |\bar{X}_{1,n}^{\leq}|^{\alpha+\eta}) \vee \sup_n (n E |\bar{X}_{1,n}^{>}|^{\alpha-\eta}) < 2^{\alpha+\eta} M' < \infty$$

and hence (3.3'), (3.4') imply again (3.2).

When  $\alpha = 1$ , we use a "mixed" proof: We define

$\bar{X}_{i,n}^{\leq}, \bar{X}_{i,n}^{>}$  as in (3.5). (This time, it is not necessary

that  $E \bar{X}_{1,n}^{>} = 0$ ). Then, we majorate  $\sup_{1 \leq k \leq m} \left| \sum_{i=1}^k b_{i,n} \bar{X}_{i,n}^{>} \right|$ , by  $\sum_{i=1}^m |b_{i,n}| \left| \bar{X}_{i,n}^{>} \right|$ , like in the case  $\alpha < 1$ ,

and apply to  $P \left\{ \sup_{1 \leq k \leq m} \left| \sum_{i=1}^k b_{i,n} \bar{X}_{i,n}^{\leq} \right| > \frac{\epsilon}{2} \right\}$  the maximal

inequality, as in the case  $\alpha > 1$ .

The case  $m = \infty$  follows trivially by letting  $m \rightarrow \infty$  in (3.3'), (3.4').  $\square$

By Lemma 2,

$$\begin{aligned} P\{|Z_n(1)| > \varepsilon\} &= P\left\{\left|\sum_{i=1}^n \sum_j c_{i-j} X_{j,n}\right| > \varepsilon\right\} \\ &= P\left\{\left|\sum_j X_{j,n} \left(\sum_{i=1-j}^{n-j} c_i\right)\right| > \varepsilon\right\} \\ &\leq \frac{M}{n} \varepsilon^{-(\alpha+n)} D_n^{(\alpha-n)}(c.) \quad , \end{aligned}$$

where

$$D_n^{(\alpha-n)}(c.) := \sum_j \left| \sum_{i=1-j}^{n-j} c_i \right|^{\alpha-n}$$

We show now that this quantity grows at most linearly in  $n$ , when  $\nu \leq 1$ .

Lemma 3: If  $\nu \leq 1$ , then, for every  $\alpha \geq \nu$  we have:

$$D_n^{(\alpha)}(c.) \leq ns(\alpha, c.)$$

Proof of Lemma 3      a) If  $\alpha \leq 1$ ,

$$D_n^{(\alpha)}(c.) = \sum_{i=-\infty}^{\infty} \left| \sum_{j=-1}^{n-1} c_j \right|^{\alpha} \leq \sum_{i=-\infty}^{\infty} \sum_{j=-1}^{n-1} |c_j|^{\alpha} = n \sum_{j=-\infty}^{\infty} |c_j|^{\alpha}.$$

If  $\alpha > 1$ ,

$$\begin{aligned} D_n^{(\alpha)}(c.) &= \sum_{i=-\infty}^{\infty} \left| \sum_{j=-1}^{n-1} c_j \right|^{\alpha} \leq \left( \sum_{j=-\infty}^{\infty} |c_j| \right)^{\alpha-\nu} \left( \sum_{i=-\infty}^{\infty} \sum_{j=-1}^{n-1} |c_j|^{\nu} \right) \\ &\leq \left( \sum_{j=-\infty}^{\infty} |c_j| \right)^{\alpha-\nu} \cdot \left( n \sum_{j=-\infty}^{\infty} |c_j|^{\nu} \right) = ns(\alpha, c.). \end{aligned}$$

Proof of Proposition 3:

$$\begin{aligned}
 \text{a) } P\{|Z_n(t_2) - Z_n(t_1)| > \varepsilon\} &= P\left\{\left|\sum_{j=1}^{[nt_2]-j} X_{j,n} \left(\sum_{i=[nt_1]-j}^{[nt_2]-j} c_i^{(n)}\right)\right| > \varepsilon\right\} \\
 &\leq \frac{M}{n} \varepsilon^{-(\alpha+r)} D_{[nt_2]-[nt_1]}^{(\alpha-r)}(c^{(n)}) \quad (\text{By Lemma 2}) \\
 &\leq M \varepsilon^{-(\alpha+r)} s(\alpha-r, c^{(n)}) \cdot \frac{[nt_2]-[nt_1]}{n} \quad (\text{By Lemma 3}) \\
 &\leq 2M \varepsilon^{-(\alpha+r)} s(\alpha-r, c^{(n)}) (t_2 - t_1)
 \end{aligned}$$

b) In the case  $\alpha \leq 1$ , we can take absolute values:

$$\begin{aligned}
 P\left\{\sup_{0 \leq t \leq 1} |Z_n(t)| > \varepsilon\right\} &\leq P\left\{\sum_{i=1}^n \frac{|X_i|}{a_n} \left(\sum_{j=1-i}^{n-i} |c_j|\right) \geq \varepsilon\right\} \\
 &\leq M \frac{\varepsilon^{-(\alpha+r)}}{n} D_n^{(\alpha-r)}(|c^{(n)}|) \quad (\text{By Lemma 2}) \\
 &\leq M \varepsilon^{-(\alpha+r)} s(\alpha-r, c^{(n)}) \quad (\text{By Lemma 3})
 \end{aligned}$$

When  $\alpha > 1$ , it follows from part a) that

$$\begin{aligned}
 P\{J_n(t_1, t, t_2) > \varepsilon\} &\quad (\text{See 2.6}) \\
 &\leq P\{|Z_n(t) - Z_n(t_1)| > \varepsilon\} \\
 &\leq L \varepsilon^{-(\alpha+r)} (t_2 - t_1) s(\alpha-r, c^{(n)})
 \end{aligned}$$

By the classical bisection method used for example in Billingsley (1968), Ex. 12.5 (see also Avram and Taqqu (1986), Proposition 1a), if we let:

$$\overline{J}_n(t_1, t_2) := \sup_{t \in [t_1, t_2]} J_n(t_1, t, t_2),$$

then

$$P\{\overline{J}_n(t_1, t_2) > \varepsilon\} \leq L k \varepsilon^{-(\alpha+r)} (t_2 - t_1) s(\alpha-r, c^{(n)}) (\ln n)^{1+\alpha+r},$$

and since

$$P\left\{\sup_t |Z_n(t)| > \varepsilon\right\} \leq P\{|Z_n(1)| > \frac{\varepsilon}{2}\} + P\{\overline{J}_n(0, 1) > \frac{\varepsilon}{2}\},$$

the result follows.

Proof of Proposition 2a. We assume w.l.o.g.  $\sum c_i \leq 1$ .

Expression (2.8) is obvious if  $[na] = [nt]$  or

$[nb] = [nt]$ . Hence, we assume that  $[nt] - [na] \geq 1$ ,

$[nb] - [nt] \geq 1$ , and note, then, that  $b - a \geq \frac{1}{n}$ . Let

$t_1$  be such that  $[nt] - [nt_1] \geq 1$ . Then,

$$Z_n(t) - Z_n(t_1) = \sum_{i=-\infty}^{\infty} X_{i,n} \sum_{k=[nt_1]-i+1}^{[nt]-i} c_k =$$

$$\sum_{i=[nt_1]-K+1}^{[nt]-K} \dots + \sum_{i=[nt]-K+1}^{[nt]+K} \dots,$$

( $c_i = 0$  for  $|i| > K$  and thus  $\sum_{k=[nt_1]-i+1}^{[nt]-i} c_k = 0$  if

$i < [nt_1]-K+1$ , or  $i > [nt]+K$ ) Letting now

$$S_1(t_1) = \sum_{i=[nt_1]-K+1}^{[nt]-K} X_{i,n} \sum_{k=[nt_1]-i+1}^K c_k,$$

$$\underline{X} = (X_{[nt]-K+1,n}, \dots, X_{[nt]+K,n}),$$

$$\underline{b}_1(t_1) = \left\{ b_1^{(i)}(t_1) \right\}_{i=[nt]-K+1}^{[nt]+K},$$

with  $b_1^{(i)}$  given by

$$b_1^{(i)}(t_1) = \sum_{k=[nt_1]-i+1}^{[nt]-i} c_k,$$

we have

$$(3.6) \quad Z_n(t) - Z_n(t_1) = S_1(t_1) + \underline{b}_1(t_1) \cdot \underline{X},$$

where the dot denotes a scalar product.

Similarly, if  $[nt_2] - [nt] \geq 1$ , then

$$(3.7) \quad Z_n(t_2) - Z_n(t) = \sum_{i=[nt]-K+1}^{[nt]+K} \dots + \sum_{i=[nt]+K+1}^{[nt_2]+K} \\ = \underline{b}_2(t_2) \cdot \underline{X} + S_2(t_2),$$

where

$$S_2(t_2) = \sum_{i=[nt]+K+1}^{[nt_2]+K} X_{i,n} \sum_{k=-K}^{[nt_2]-i},$$

and

$$\underline{b}_2(t_2) = \left\{ b_2^{(i)}(t_2) \right\}_{i=[nt]-K+1}^{[nt]+K}$$

with

$$b_2^{(i)}(t_2) = \sum_{k=[nt]-i+1}^{[nt_2]-i} c_k.$$

The decompositions (3.6), (3.7) are such that  $\underline{X}$ ,  $S_1(t_1)$

and  $S_2(t_2)$  are independent.

Since oscillations of type  $M$  are zero when they go in the same direction, we have

$$P \left\{ w_{[a,t,b]}^M(Z_n) \geq \epsilon \right\} \\ \leq P \left\{ \sup_{a \leq t_1 \leq t} S_1(t_1) + \underline{b}_1(t_1) \cdot \underline{X} \geq \epsilon, \inf_{t \leq t_2 \leq b} S_2(t_2) + \underline{b}_2(t_2) \cdot \underline{X} \leq -\epsilon \right\} \\ + P \left\{ \inf_{a \leq t_1 \leq b} S_1(t_1) + \underline{b}_1(t_1) \cdot \underline{X} \leq -\epsilon, \sup_{t \leq t_2 \leq b} S_2(t_2) + \underline{b}_2(t_2) \cdot \underline{X} \geq \epsilon \right\}.$$

We shall estimate each term separately, and since the proofs are similar, we consider only the first term.

Consider the following events:

$$S_1 = \left\{ \sup_{a \leq t_1 \leq t} S_1(t_1) \geq \frac{\epsilon}{2} \right\}$$

$$S_2 = \left\{ \inf_{t \leq t_2 \leq b} S_2(t_2) \leq -\frac{\epsilon}{2} \right\}$$

$$X_1 = \left\{ \sup_{a \leq t_1 \leq t} b_1(t_1) \cdot \tilde{X} \geq \frac{\epsilon}{2} \right\}$$

$$X_2 = \left\{ \inf_{t \leq t_2 \leq b} b_2(t_2) \cdot \tilde{X} \leq -\frac{\epsilon}{2} \right\}$$

$$A_1 = \left\{ |X_{1,n}| \leq \frac{\epsilon}{8K} \right\}, \quad 1 = [nt] - K + 1, \dots, [nt] + K$$

$$E = \left\{ \sup_{a \leq t_1 \leq t} S_1(t_1) + b_1(t_1) \cdot \tilde{X} \geq \epsilon, \inf_{t \leq t_2 \leq b} S_2(t_2) + b_2(t_2) \cdot \tilde{X} \leq -\epsilon \right\}$$

Then

$$E \subset (S_1 \cup X_1) \cap (S_2 \cup X_2)$$

and hence

$$\begin{aligned} P(E) &\leq P(S_1)P(S_2) + P(S_1)P(X_2) \\ &\quad + P(S_2)P(X_1) + P(X_1 \cap X_2). \end{aligned}$$

We explain now the idea of the proof: each of  $P(S_1)$ ,  $P(S_2)$ ,  $P(X_1)$ ,  $P(X_2)$  ought to be  $\approx 0(t_2 - t_1)$ , by Proposition 3a, and thus their products will be  $0(t_2 - t_1)^2$ , which enables one to use the "bisection" method (see Billingsley (1968), Theorem 12.1). The only term which may cause difficulties is  $P(X_1 \cap X_2)$ . However, we expect all the components of  $\tilde{X}$ , except at most 1, to be negligible, and since the coefficients are

non-negative, we cannot have at the same time

$$\underline{b}_1(t_1) \cdot \underline{X} > \frac{\epsilon}{2}, \quad \underline{b}_2(t_2) \cdot \underline{X} < -\frac{\epsilon}{2}, \quad \text{and so the}$$

event  $X_1 \cap X_2$  should be negligible. (It is possible to see here why  $J_1$  convergence may not work. If

$$|\underline{b}_1(t_1) \cdot \underline{X}| > \frac{\epsilon}{2}, \quad \text{then we can also have } |\underline{b}_2(t_2) \cdot \underline{X}| > \frac{\epsilon}{2},$$

and thus if we attempt to compute the probability that the

$J_1$  oscillation is bigger than  $\epsilon$ , we would get that

$$P\left\{|\underline{b}_1(t_1) \cdot \underline{X}| > \frac{\epsilon}{2}, |\underline{b}_2(t_2) \cdot \underline{X}| > \frac{\epsilon}{2}\right\} \text{ is the dominant term,}$$

and thus merely  $O(t_2 - t_1)$ .

Note, now, that

$$P(A_i^C) = P\left\{|\underline{X}_{i,n}| > \frac{\epsilon}{8K}\right\} \leq \frac{M\left(\frac{\epsilon}{8K}\right)}{n}^{-(\alpha+\eta)}$$

(Apply Lemma 2 with  $b_{i,n} = 1$ ,  $m = 1$ .)

Next, note that  $\bigcap_{i=[nt]-K+1}^{[nt]+K} A_i \subset X_1^C$ , since then

$$|\underline{b}_1(t_1) \cdot \underline{X}| \leq \sum_1^{[nt]+K} b_1^{(i)}(t_1) \frac{\epsilon}{8K} \leq \sum_1^{\infty} \frac{\epsilon}{8K} \leq \frac{\epsilon}{4}.$$

Hence  $X_1 \subset \bigcup_1 (A_i)^C$

and

$$\begin{aligned} (3.8a) \quad P(X_1) &\leq 2K P\left\{|\underline{X}_{i,n}| \geq \frac{\epsilon}{8K}\right\} \\ &\leq 2K \cdot \frac{M\left(\frac{\epsilon}{8K}\right)}{n}^{-(\alpha+\eta)} = \frac{M'\epsilon^{-(\alpha+\eta)} K^{1+\alpha+\eta}}{n}. \end{aligned}$$

Similarly,

$$(3.8b) \quad P(X_2) \leq \frac{M'\epsilon^{-(\alpha+\eta)} K^{1+\alpha+\eta}}{n}.$$

We now show that  $P(X_1 \cap X_2)$  is small. Note first that

$$(3.9) \quad \bigcap_{i \neq i_0} A_i \cap A_{i_0}^C \cap X_1 \cap X_2 = \phi.$$

Indeed, suppose  $X_{i_0} > 0$ . Then the L.H.S. of (3.9) is contained in  $\bigcap_{i \neq i_0} A_i \cap A_{i_0}^C \cap X_1$ , and if that event is not empty, there is a  $t_2$  such that

$$\begin{aligned} -\frac{\epsilon}{2} &> b_2(t_2) \cdot X = b_2^{(i_0)}(t_2) X_{i_0, n} + \sum_{i \neq i_0} b_2^{(i)}(t_2) X_{i, n} \\ &> b_2^{(i_0)}(t_2) \frac{\epsilon}{8K} - \frac{\epsilon}{4}, \end{aligned}$$

contradicting  $b_2^{(i_0)}(t_2) \geq 0$ . A similar argument holds if  $X_{i_0} \leq 0$ .

Thus if  $X_1 \cap X_2$  occurs, it is not possible that exactly one  $A_i^C$  occurs. It is also easy to check that some  $A_i^C$  must occur. Therefore,  $A_i^C$  must occur for at least two different  $i$ 's. Hence

$$\begin{aligned} P(X_1 \cap X_2) &\leq \sum_{i_0 \neq i_1} P[A_{i_0}^C \cap A_{i_1}^C] \\ &\leq K^2 (P[A_i^C])^2 \\ &\leq K^2 \left[ \frac{M \left( \frac{\epsilon}{8K} \right)}{n} \right]^{-(\alpha + \eta)} \\ (3.10) \quad &= M'' K^{2(1 + \alpha - \eta)} \frac{\epsilon^{-2(\alpha + \eta)}}{n^2}. \end{aligned}$$

We now turn to  $S_1$  and  $S_2$  and show that

$$(3.11a) \quad P(S_1) \leq M e^{-(\alpha+\eta)}(t-a)$$

and

$$(3.11b) \quad P(S_2) \leq M e^{-(\alpha+\eta)}(b-t)$$

where  $M$  is again a generic constant.

Let

$$S_1^M = \max_{[na]-K+1 \leq l \leq [nt]-K} \sum_{i=1}^{[nt]-K} X_{i,n}$$

$$S_2^m = \min_{[nt]+K+1 \leq \ell \leq [nb]+K} \sum_{i=[nt]+K+1}^{\ell} X_{i,n}.$$

Note that

$$(3.12a) \quad S_1(t_1) = \sum_{k=K+1-([nt]-[nt_1])}^K c_k \sum_{i=[nt_1]-k+1}^{[nt]-K} X_{i,n}$$

$$\leq \left[ \sum_{k=-K}^K c_k \right] S_1^M 1_{\{S_1^M > 0\}} \leq S_1^M 1_{\{S_1^M > 0\}}$$

and, similarly,

$$(3.12b) \quad S_2(t_2) = \sum_{k=-K}^{[nt_2]-[nt]-K-1} c_k \sum_{i=[nt]+K+1}^{[nt_2]-k} X_{i,n}$$

$$\geq \left[ \sum_{k=-K}^K c_k \right] S_2^m 1_{\{S_2^m < 0\}} \geq S_2^m 1_{\{S_2^m < 0\}}$$

Applying Lemma 2, with  $b_{1,n} = 1$ ,  $m = [nt] - [na]$ , and  $m = [nb] - [nt]$ , respectively, we get

$$\begin{aligned}
P\left[S_1(t_1) > \frac{\epsilon}{2}\right] &\leq P\left[S_1^M \Big|_{\{S_1^M > 0\}} > \frac{\epsilon}{2}\right] \\
&\leq P\left[S_1^M > \frac{\epsilon}{2}\right] \\
&\leq M\left[\frac{\epsilon}{2}\right]^{-(\alpha+\eta)} \frac{[nt] - [na]}{n} \\
&\leq 2M\left[\frac{\epsilon}{2}\right]^{-(\alpha+\eta)} (t-a),
\end{aligned}$$

and similarly,

$$P\left[S_2(t_2) < -\frac{\epsilon}{2}\right] \leq 2M\left[\frac{\epsilon}{2}\right]^{-(\alpha+\eta)} (b-t).$$

This establishes (3.11a) and (3.11b).

Putting together (3.8), (3.10), (3.11) and introducing a new constant  $L$ , we get

$$P(E) \leq L \epsilon^{-2(\alpha+\eta)} \left[ (b-a)^2 + 2(b-a) \frac{K^{1+\alpha+\eta}}{n} + \frac{K^{2(1+\alpha+\eta)}}{n^2} \right].$$

Now  $\frac{1}{n} < b-a$  (otherwise the oscillations are zero),  
 $b-a < 1$  and  $n^{(1/2-\eta)/(1+\alpha+\eta)} > K$ . Therefore,

$$\frac{K^{1+\alpha+\eta}}{n} < \frac{1}{n^{1/2+\eta}} < (b-a)^{(1/2)+\eta}.$$

Since  $b-a \leq 1$ , we get

$$\begin{aligned}
P(E) &\leq L \epsilon^{-2(\alpha+\eta)} \left[ (b-a)^2 + 2(b-a)^{3/2+\eta} + (b-a)^{1+2\eta} \right] \\
&\leq 4 L \epsilon^{-2(\alpha+\eta)} (b-a)^{1+2\eta}.
\end{aligned}$$

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